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Jet Development in Leading Log QCD*

by

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ABSTRACT

A simple picture of jet development in QCD is described. Various applications are treated, including transverse spreading of jets, hadroproduced γ^* p_T distributions, lepton energy spectra from heavy quark decays, soft parton multiplicities and hadron cluster formation.

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According to QCD, high-energy e^+e^- annihilation into hadrons is initiated by the production from the decaying virtual photon of a quark and an antiquark, each with invariant masses up to the c.m. energy $\sqrt{s}$ in the original e^+e^- collision. The q and $\bar{q}$ then travel outwards radiating gluons which serve to spread their energy and color into a jet of finite angle. After a time $\gtrsim 1/\sqrt{s}$, the rate of gluon emissions presumably decreases roughly inversely with time, except for the logarithmic rise associated with the effective coupling constant, ($\alpha_s(t) \sim 1/\log(t/\Lambda^2)$, where $\sqrt{t}$ is the invariant mass of the radiating quark). Finally, when emissions have degraded the energies of the partons produced until their invariant masses fall below some critical $\sqrt{t_c}$ (probably a few times Λ), the system of quarks and gluons begins to condense into the observed hadrons.

The probability for a gluon to be emitted at times of $O(\frac{1}{\sqrt{s}})$ is small and may be estimated from the leading terms of a perturbation series in $\alpha_s(s)$. Any gluon produced at these early times will typically be at a large angle to the q, $\bar{q}$ directions (so that the jet it initiates is resolved) and will have an energy $\sim \sqrt{s}$: thus the wavelength of a gluon 'emitted from q' encompasses $\bar{q}$, so that interferences between the various amplitudes for gluon emissions are important. At times $\gg 1/\sqrt{s}$, the average total number of emitted gluons grows rapidly (see eq. 9) with time, and one must sum the effects of many gluons radiated at progressively smaller angles, but with energies $\sim \sqrt{s}$. Usually the wavelength of one radiated gluon does not reach the point at which the last was emitted, and hence at these times the sequence of gluon emissions in a jet may be treated independently from each other and from those in other jets. Below I shall mostly discuss the development of jets in this semiclassical regime, where the leading log approximation (LLA) may be used: some details of the results are contained in Refs. [1] and [2]. The ultimate transformation of the quarks and gluons in each jet into hadrons (which undoubtedly involves consideration of amplitudes, rather than probabilities) is quite beyond any perturbative methods, but, at least locally, depends only on the energy and quantum numbers of a jet, and not on the details of the process by which the jet was produced (except perhaps because of low-energy remnants from initial hadrons or nuclei). (The formation of a jet from an off-shell quark in many respects parallels the development of an electromagnetic shower from a high-energy electron in matter, for which the probabilistic LLA is accurate above a fixed critical energy below which ionization losses dominate.)

The times and distances quoted here are in the rest frame of the radiating quark. In the c.m. frame, they are dilated by $\gamma = E/E_0 \sim \sqrt{s/t}$. A parton off-shell an amount $\sqrt{t}$ should typically survive a time $\tau \sim 1/\sqrt{t}$ (this is clear on dimensional grounds or from the energy denominators $\Delta E \sim 1/\tau \sim E - |\vec{p}|$ in non-covariant perturbation theory). A system of partons apparently forms hadrons when the parton invariant masses $\sqrt{t} \sim \sqrt{t_c} \sim \Lambda$, corresponding to a distance $\sim \sqrt{s}/\Lambda^2$ in the c.m.s. (at this distance a string with $\kappa \sim \Lambda^2$ stretched between the q, $\bar{q}$ would have dissipated their original kinetic energy). Note that if confinement acted at a fixed time $\sim 1/\Lambda$ in the c.m.s., then $t_c \sim \Lambda\sqrt{s}$, and no scaling violations should occur in fragmentation functions (since $\log(s/\Lambda^2)/\log(t_c/\Lambda^2)$ is independent of s). (Such a mass would result from rescattering of a parton with $E \sim \sqrt{s}$ from a cloud of low energy partons with momenta $\sim \Lambda$: in e^+e^- annihilation, such a cloud forms only at $t_{cms} \gtrsim \sqrt{s}/\Lambda^2$, but in hadron reactions such spectators may seriously affect the structure of the final state.) The time of hadron formation may be investigated directly in collisions with nuclei: if $t_c \sim \Lambda^2$ then partons produced within a nucleus should form hadrons only far outside it, in a manner uninfluenced by its presence [F.1].

One approach in studying QCD jet development is to consider quantities which are insensitive to all but the short time region described by low-order perturbation theory. The simplest such observable is the total cross-section for e^+e^- annihilation to hadrons. QCD corrections modify the wavefunctions for the q, $\bar{q}$ even at the moment of production, and thereby correct the Born term. Attractive one-gluon exchange at short distances enhances the cross-section by a factor $1 + \alpha_s(s)/\pi$ [F.2], while the effects of the eventual confinement of the quarks (at short distances similar to the acquisition of an effective mass) are suppressed by an energy denominator to be $O(\Lambda^2/s)$. (Close to heavy $Q\bar{Q}$ production thresholds, the $Q,\bar{Q}$ have long wavelengths ($\sim 1/(m_Q v)$), and their wavefunctions are therefore sensitive to interactions at large times: such threshold regions must simply be smeared over.) In processes involving initial hadrons (e.g., $\gamma^*N \to X$), only scatterings which deflect initial partons outside the cylinders (of fixed transverse dimension $1/\sqrt{t_c^i} \sim 1/\Lambda$) formed by the incoming hadrons contribute to observable cross-sections. Just before a scattering involving momentum transfer Q, gluons will typically be emitted with differential cross-section $\sim dk_T/k_T$ up to $k_T \simeq Q$. The probability for gluon emission (which affects the cross-section by 'spreading' the initial parton) outside the initial cylinder $\sim \log(Q^2/t_c^i)$; because the size $1/\sqrt{t_c^i}$ of the initial hadron is fixed with Q^2, such terms give rise to 'scaling violations' which cause the cross-section to depend on Q^2/t_c^i. For a given initial hadron, the terms divergent as its size is taken to infinity are known to be universal and independent of the details of the parton scattering [4]; they are determined by processes which act at large times before the interaction.

One may obtain further information on the short distance structure of QCD processes from the angular distributions of hadronic energy in their final states. (A convenient set of shape parameters for this purpose is the $H_\ell = \sum_{i,j} E_i E_j/s\ P_\ell(\cos\phi_{ij})$ [5].) If in studying final states, hadrons with low energies are ignored and sets of hadrons separated by angles less than, say θ, are lumped together into 'jets', then the lumped energy distributions are typically sensitive to the structure of events only at times $\lesssim 1/(\theta\sqrt{s})$, since particles radiated later will usually not be 'resolved'. (In the $\langle H_\ell \rangle$, the behavior of the Legendre polynomials implies $\theta \sim 1/\ell$.) Nevertheless, it turns out that the residual effects of confinement at large distances are more important for shape parameters than for total cross-sections: they suffer $O(\Lambda/\sqrt{s})$ rather than $O(\Lambda^2/s)$ corrections [F.3]. As θ is decreased, measures of final state energy distributions become progressively more sensitive to nearly collinear emissions occurring with high probability, typically at times $\sim 1/(\theta\sqrt{s})$.

In diagrammatic calculations, the approximate independence of small transverse momentum gluon emissions from the q and $\bar{q}$ produced in e^+e^- annihilation (or the incoming and outgoing q in $\gamma^*q \to X$, etc.) is best revealed by using axial gauges $n.A = 0$ for the gluon propagator. In these gauges, interference terms are suppressed, and a probabilistic interpretation of single (ladder) diagrams is possible. The choice of n determines what fraction of the radiation appears to come from each of the quarks: if n is chosen symmetrically with respect to their momenta then they appear to radiate equally; if n is along one quark direction, then the gluons appear to come from the other quark, although some travel backwards with respect to its momentum. In a suitable gauge, the differential cross-section for emissions of k low transverse momentum gluons from an incoming or outgoing quark may be written in the simple product form [6,7]

$$\frac{d\sigma}{dz_1\ldots dz_k\, dt_1\ldots dt_k} \simeq \left[\frac{P_{qq}(z_1)}{2\pi}\,\frac{\alpha_s(t_1)}{\hat{t}_1}\right]\ldots$$

$$\ldots\left[\frac{P_{qq}(z_k)}{2\pi}\,\frac{\alpha_s(t_k)}{\hat{t}_k}\right], \tag{1}$$

$$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)_+$$

$$\int_0^1 (h(z))_+ f(z)dz = \int_0^1 h(z)(f(z)-f(1))dz,$$

where z_i is the relative longitudinal Sudakov variable (roughly energy fraction) of the $(i+1)^{th}$ quark with respect to the i^{th} quark ($z_i = (p^o_{i+1}+p^3_{i+1})/(p^o_i+p^3_i)$; 3 along $\vec{p}_i$) and t_i is the invariant mass of the i^{th} quark link ($\hat{t}_i = t_i/s$). The terms dropped from the leading log approximation (1) contain extra $\hat{t}_i$ factors; these may only be neglected if $\hat{t}_i << 1$ (although $t_i >> \Lambda^2$ for confinement effects to be ignored). Kinematics require that $t_i \le t_{i-1}$, $0 \le z_i \le 1$. Many consequences of (1) follow simply from integrating over more restricted phase space volumes so as to select only jets obeying various criteria. In addition to radiation of real gluons, (1) includes virtual gluon corrections to quark lines or to vertices which contribute leading log terms at the points $z_i = 1$, $t_i = t_{i-1}$. If the external kinematic constraints imposed allow such diagrams to contribute (so that z_i integrals run right up to 1), then the $\int dz/(1-z)$ [F.4] divergences from the soft gluon emissions are canceled by the virtual diagrams. (The remaining infrared divergences, apparent at small t, arise from emission of hard gluons collinear to a massless quark and are cut off by the finite propagation time of the quark, implemented in perturbation theory by exchanges with other jets or by the effects of the cylinders of initial partons representing hadrons.) The contribution of a virtual correction to a quark link of mass $\sqrt{t}$ may be written (by introducing Sudakov variables into the internal loop integration) roughly as $\delta(1-z) \int_0^1 (1+z'^2)/(1-z')dz' \int_{t_{min}}^{t} dt'/t'$, where the internal t' integration is cut off by the same large distance effects as are the external t integrations, so that $t_{min} \simeq t_c$. For most applications, the virtual diagrams may then be included as in (1) simply by adding a divergent $-\delta(1-z)$ term to $P_{qq}(z)$ (hence the +); then the $\log(t/t_c)$ from internal loop integration will be reproduced by integration over the external t. This procedure will be sufficient so long as t_i is allowed to run up to t_{i-1} whenever z_i runs up to 1 (so that virtual diagrams contribute). (This will certainly be the case if the k_T, but not angles of emitted gluons are considered.)

The formula (1) accounts only for gluon emissions from the original quark: to describe radiation from the gluons produced, one must append similar product forms, with appropriate P_{qq} replaced by P_{GG}, P_{qG} or P_{Gq} [F.5] according to the type of emission. In many calculations, one is concerned with the behavior of only one or two partons, and in this case, one need essentially consider only the possible 'backbones' of the jet, which connect the initial parton to the partons considered (provide their structural support in the tree); further emissions from partons not in the backbone may be disregarded, since integrating their contributions to the cross-section over available phase space simply gives a factor one. To describe the production of the partons considered, one must sum over all possible backbones and integrate over the ordered t_i of the partons along them. The differential cross-section for a given backbone consisting of k partons i_1, i_2, i_3.... involves the product $P_{i_1 i_2}(z_1)P_{i_2 i_3}(z_2)$.... . When the required integrals of this are summed over k, they often form an exponential series, in which the exponent contains a matrix of (the z^n moments of) the $P_{ij}(z)$; ordered expansion of the matrix exponential accounts for all possible backbones with the correct combinatorial weights.

As a first application of eq. (1), I estimate the mean product of energies incident on two back-to-back detectors of angular size θ around an e^+e^- event. For the lowest-order process, $e^+e^- \to q\bar{q}$, the energy correlation $\$_B(\theta) \equiv 2\langle E_i E_{i'}\rangle/s$ (E_i is the energy incident per unit area on detector i; i' is antipodal to i) is 1 (for $\theta \neq 0$): if q enters one detector, $\bar{q}$ must be incident on the other. (For large ℓ, $\langle H_\ell\rangle \simeq [\langle E_i^2\rangle+(-1)^\ell\langle E_i E_{i'}\rangle]/(2s)$, where $\theta \simeq 1/\ell$; in dominantly two-jet processes $\langle E_i^2\rangle \ll \langle E_i E_{i'}\rangle$.) $\$_B(\theta)$ deviates from one when gluon emissions deflect energy outside angle $\simeq \theta$ cones around the $q,\bar{q}$ directions. To LLA, the energies of radiated gluons are negligible; their only effect is to deflect the original $q,\bar{q}$: $\$_B(\theta)$ thus becomes simply the total probability that the final $q,\bar{q}$ should have transverse momenta $k_T \leqslant \theta\sqrt{s}$. The i^{th} gluon emission imparts a $(k_T^2)_i = (1-z_i)(z_i t_i - t_{i+1}) \simeq (1-z_i)t_i$ to the quark. $\$_B(\theta)$ is the integral of the differential cross-section (1) (summed over all possible numbers of emissions) subject to the constraint $\Sigma(k_T^2)_i \leqslant \theta^2 s$; all radiated gluons must therefore be both soft ($(1-z_i)$ small) and nearly collinear to the quarks (t_i small). The necessary integrals are most conveniently calculated by subtracting from one those obtained by integrating outside the constraints $(1-z_i)t_i \leqslant \theta^2 s$. (In this way, one need only consider real emissions and is not concerned with delicate cancellations from virtual processes.) Consider first the emission of one gluon. To satisfy $k_T < \theta\sqrt{s}$, z_1 must be integrated from $\simeq 0$ only up to $\sim 1 - \theta^2 s/t_1$, rather than 1. The $1/(1-z_1)$ soft divergence in $P_{qq}(z_1)$ thus contributes a term $\sim \log(t_1/\theta^2 s)$ [F.6]. Integrating over t_1 from $\sim \theta^2 s$ to $\simeq s$ gives the final $O(\alpha_s)$ result $\$_B(\theta) \simeq 1 - \frac{8\alpha_s}{3\pi}\log^2\theta$. Notice that the variation of $\alpha_s(t_1 \sim 1/\log(t_1/\Lambda^2)$ over the range of the t_1 integration must be ignored to leading log accuracy compared to the $\log(t_1)$ result from the z_1 integral: its effects are formally of the same order as other subleading log corrections, which change the scale of the θ in the final result [F.7]. While the leading log terms are independent of the process by which the $q,\bar{q}$ were produced, the subleading logs are not universal. In the leading log approximation, the gluon emissions are all independent, except for the phase space restriction $t_i \leq t_{i-1}$. Hence the contribution to $\not{s}_B(\theta)$ from k gluon emission $\simeq (-2\alpha_s/3\pi)^k \log^{2k}(\theta^2)/k!$: the crucial $1/k!$ arises from the nesting of the t_i integrations. Summing over k then gives [F.8]

$$\not{s}_B(\theta) \simeq \exp[-\frac{8\alpha_s}{3\pi}\log^2\theta]. \tag{2}$$

In contrast to the $O(\alpha_s)$ result, this form vanishes as $\theta \to 0$, reflecting the fact that the $q,\bar{q}$ will always be at least slightly deflected by radiation. However, at the small θ ($\lesssim \exp(-1/\alpha_s)$) where the leading log eq. (2) is damped, thus far uncalculated subleading log effects will probably dominate: when $\theta^2 \lesssim \Lambda^2/s$ (i.e., $\ell \gtrsim \sqrt{s}/\Lambda$ for $\langle H_\ell \rangle$), (2) must fail, since then the emissions no longer occur before hadronization. (Phenomenological simulations of hadron formation suggest that, in practice, perturbative results become inaccurate at much larger angles.) Note that if the variation of $\alpha_s(t)$ had been retained in the t_i integrals, (2) would become ($\alpha_s(t) = \beta_o/\log(t/\Lambda^2)$)

$$\begin{aligned}\not{s}_B(\theta) &\simeq \exp[-\frac{4\beta_o}{3\pi}(\log(1+\frac{\log(\theta^2)}{\log(s/\Lambda^2)})\log(\frac{\theta^2 s}{\Lambda^2}) - \log(\theta^2))] \\ &\simeq \exp[-\frac{8\alpha_s}{3\pi}\log^2\theta(1-\frac{\log\theta}{\log(s/\Lambda^2)}+\ldots)];\end{aligned} \tag{3}$$

the change cannot consistently be kept in the LLA.

Equation (2) gives approximately the probability that a produced or struck quark emits no gluons with total $k_T \geqslant \theta\sqrt{s}$ and, therefore, typically propagates without radiation for a time $\gtrsim 1/\theta\sqrt{s}$. It is thus similar to the quark (Sudakov) form factor, which gives roughly the probability that no gluons are emitted before a time $\sim 1/p^2$, where $\sqrt{p^2}$ is a (regularizing) invariant mass assigned to final quarks and/or gluons. (In the Sudakov form factor, θ^2 is roughly replaced by $(p^2/s)^\lambda$; where λ depends on the precise method of regularization used [F.9].)

The results obtained above may be applied directly to estimate the transverse momentum (p_T) spectrum of virtual photons (γ^*) produced in hadron collisions. The leading log terms come from the process in which a q and a $\bar{q}$ from the initial hadrons suffer transverse deflections by the emission of gluons (but retain roughly their original energy) before annihilating to the γ^*. Then the p_T spectrum is obtained from the (derivative of the) deflection probability (2) as [9]

$$\frac{1}{\sigma_o}\frac{d\sigma}{dp_T^2} \simeq -\frac{4\alpha_s}{3\pi p_T^2}\log\left(\frac{p_T^2}{s}\right)\exp\left[-\frac{2\alpha_s}{3\pi}\log^2\left(\frac{p_T^2}{s}\right)\right] \tag{4}$$

where σ_o is the cross-section without gluon emissions, and $\sqrt{s}$ is roughly the invariant mass ($\sqrt{Q^2}$) of the γ^* (which is formally indistinguishable from the incoming $q\bar{q}$ c.m. energy $\sqrt{s}$ in the LLA). However, as with eq. (2), this result is rarely adequate. At large p_T ($\sim\sqrt{s}$), the exact $O(\alpha_s)$ p_T spectrum (including subleading log terms not accounted for in (4)) should be sufficient; at small p_T higher-order terms could potentially be significant, but the leading logs of (4) are damped at small p_T and so may be overwhelmed by subleading log corrections (for $p_T^2 \leqslant \sqrt{s\Lambda^2}$). (Subleading logs from hard, but collinear (small t), emissions may be accounted for by keeping the full P(z) in the derivation of (4), rather than approximating $P_{qq}(z) \sim \log(1-z_{max})\delta(1-z)$; this yields a more complicated form in the exponent of (4), which is a convolution sampling $Q^2/S < 1$, and thus not simply a multiplicative correction to σ_o. Subleading logs from soft (but non-collinear) emissions plausibly exponentiate as in massive QED.) In practice, p_T must be measured with respect to the incoming hadrons rather than the $q,\bar{q}$, introducing a further spread in p_T^2 of order $t_c^i \sim \Lambda^2$.

A significant fraction of hadroproduced $^3S_1$ $Q\bar{Q}$ states (e.g., Υ; denoted here generically by ζ) probably arises from decays $\chi \to \zeta\gamma$ of even-spin χ produced by GG 'annihilation'. The resulting ζ p_T spectrum is given in LLA by replacing $4/3(= C_F)$ in eq. (4) with $3(= C_A)$ and is, therefore, broader than for γ^*, at least for $\Lambda^2 \ll p_T^2 \ll s$, m_ζ^2.

For deep-inelastic scattering, similar analysis shows that in the LLA, the distribution of final transverse momenta with respect to the γ^* direction (i.e., $\sum_i |p_T^i| = C_o$) should follow roughly the form (4) (in this approximation, only the q energy is significant). It is interesting to speculate on the differences between the p_T spectra in deep-inelastic scattering and the Drell-Yan process. While $p_T^2 > 0$ always, $s > 0$ for Drell-Yan but $s < 0$ for deep-inelastic scattering. Thus one might expect a subleading log difference between the integrated spectra by a large factor, perhaps $\sim \exp(2\pi\alpha_s/3)$.

In muon decay, the outgoing electron spectrum close to the endpoint $x = 2E_e/m_\mu \simeq 1 - O(m_e^2/m_\mu^2)$ is softened by emission of many low k_T photons. In the LLA, and taking $m_e = 0$, the methods used to derive (2) give the approximate formula

$$\frac{d\Gamma}{dx} \simeq \frac{d\Gamma_o}{dx} \exp(-\frac{\alpha}{2\pi} \log^2(1-x)), \tag{5}$$

which is independent of the details of the decay; for μ decay, $d\Gamma_o/dx = 2x^2(3-2x)$. (If $m_e \neq 0$, then divergences from photons emitted nearly collinear to the e are regulated, leaving only those from soft photons and replacing $\log^2(1-x)$ by $2\log(m_e^2/m_\mu^2)\log(1-x)$: in this case, all subleading $\log(1-x)$ terms are known also to exponentiate.) Different $d\Gamma_o/dx$ cannot be distinguished in the LLA. With $d\Gamma_o/dx = 2x^2(3-2x)$ ($\mu \to eX$ (or $b \to \ell X$) spectrum), the $O(\alpha)$ term in the expansion of (5) implies a correction to the total decay rate of $(1-265/144\ \alpha/\pi) \simeq (1-1.84\ \alpha/\pi)$; with $d\Gamma_o/dx = 12x^2(1-x)$ ($\mu \to \nu_\mu X$ (or $C \to \ell X$) spectrum) $\Gamma/\Gamma_o \doteq (1-0.8\ \alpha/\pi)$ and for $d\Gamma_o/dx = 1$, $\Gamma/\Gamma_o \simeq (1-\alpha/\pi)$; the exact result for V-A μ decay is [10] $(1-(\pi^2-25/4)\alpha/(2\pi)) \simeq (1-1.81\ \alpha/\pi)$. One may guess the correction to Γ summed to all orders in α by integrating just the LLA (5), which yields (taking $d\Gamma_o/dx = 1$):

$$\Gamma \simeq \Gamma_o \frac{\pi}{\sqrt{2\alpha}} e^{\pi/2\alpha} \mathrm{erfc}\left(\sqrt{\frac{\pi}{2\alpha}}\right),$$
$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2} dx. \tag{6}$$

For $\alpha/\pi = 0.1$, this gives $0.87\ \Gamma_o$ compared to the $O(\alpha)$ result $0.8\ \Gamma_o$, while for $\alpha/\pi = 0.4$, it gives $0.79\ \Gamma_o$ compared to $0.6\ \Gamma_o$. These results for μ decay may also be applied to the lepton energy spectra and rates for semileptonic decays $Q \to q'\ell\nu$ of heavy quarks [11], after the substitution $\alpha \to 4\alpha_s/3$. In charm decays, $O(m_s/m_c)$ and $O(\Lambda/m_c)$ effects still dominate over $O(\alpha_s)$ ones, but for b decays QCD corrections should be relevant. Here (5) gives QCD corrections to the lepton spectrum from the weak decay of an on-shell massive quark. If, as in e^+e^- annihilation, the initial Q is produced off its mass shell, gluon emissions degrade its energy by a factor $\sim [\alpha_s(s)/\alpha_s(m_Q^2)]^{0.4}$ long before the weak decay occurs (m_Q acts as a cutoff for collinear hard gluon emissions).

A similar analysis gives the modification of the γ energy spectrum from $\zeta \to \gamma GG\ldots$ due to radiation from the outgoing gluons as $\sim \exp(-3\alpha_s/(2\pi) \times \log^2(1-x))$. Integrating this over x (with $d\Gamma_o/dx$) suggests the rash guess $\Gamma \simeq (1-4.5 \frac{\alpha_s}{\pi})\Gamma_o$.

The momenta of partons produced in e^+e^- annihilation should lie roughly in a plane; deviations from coplanarity may be measured by $\Pi_1 \equiv \sum_{i,j,k} (\vec{p}_i x\vec{p}_j \cdot \vec{p}_k)/(\sqrt{s})^3(\hat{p}_i x\hat{p}_j \cdot \hat{p}_k)$ [5] ($\Pi_1 = 0$ for coplanar events and $\Pi_1 = 2/9$ for isotropic final states). The lowest-order contribution to $<\Pi_1>$ in e^+e^- annihilation is from $e^+e^- \to q\bar{q}GG$ (for which $\Pi_1^2 \sim (1-z_1)(1-z_2)(t_1t_2/s^2)^2$), and in the LLA this gives $1/\sigma\, d\sigma/d\Pi_1 \simeq 8/9(\alpha_s/\pi)^2|\log^3\Pi_1|/\Pi_1$ at small Π_1 ($e^+e^- \to q\bar{q}q'\bar{q}'$ gives only an $O(\log\Pi_1/\Pi_1)$ term). In ζ decays, $<\Pi_1>$ is larger; $\zeta \to GGGG$ (which is allowed, unlike the analogous positronium decay, as a direct consequence of the non-Abelian nature of the G couplings) gives $1/\sigma\, d\sigma/d\Pi_1 \simeq 3(\alpha_s/\pi)|\log\Pi_1|/\Pi_1$ ($\zeta \to GGq\bar{q}$ contributes $O(1/\Pi_1)$). In both cases, the integrated Π_1 distributions exponentiate when summed to all orders in α_s.

Now consider the energy correlation $\mathcal{S}_F(\theta) = 2<E_i^2>/s$ which gives the mean square energy in a jet concentrated within a cone of angle θ. Whereas $\mathcal{S}_B(\theta)$ contained $[\alpha_s\log^2\theta]^k$ terms, only $[\alpha_s\log\theta]^k$ appears in $\mathcal{S}_F(\theta)$. (For large ℓ, $<H_\ell> \simeq [\mathcal{S}_F(1/\ell) + (-1)^\ell\mathcal{S}_B(1/\ell)]/2$; only for dominantly two-jet processes (e.g., $e^+e^- \to q\bar{q}\ldots$) is $\mathcal{S}_B$ significant: when the lowest order involves > 2 final partons (as in $\zeta \to GGG$), $\mathcal{S}_F$ determines $<H_\ell>$). The deviations of $\mathcal{S}_F(\theta)$ from one are dominated by radiations in which the emitted and recoiling parton make an angle > θ. To LLA, this angle is simply t_i/s, where $\sqrt{t_i}$ is the invariant mass of the radiating parton. Here the crucial difference between $\mathcal{S}_F$ and $\mathcal{S}_B$ becomes apparent: a given emission will not affect $\mathcal{S}_F(\theta)$ so long as its products are collinear to within an angle $\simeq \theta$; however, in $\mathcal{S}_B(\theta)$ they must rather have a relative transverse momentum $\lesssim \theta\sqrt{s}$ and thus be not only almost collinear (small t), but the radiated parton must also be soft (small 1-z). The greater restriction of phase space in the latter case forbids complete cancellation of soft gluon emission divergences and leads to double rather than single log terms. Note that because of the ordering of the t_i ($t_i \gg t_{i+1}$), the dominant contributions to $\$_F(\theta)$ come from the first few emissions; subsequent radiations must have much smaller angles and therefore will not spread jets sufficiently to affect $\$_F(\theta)$. On the other hand, $k_T^2 \sim (1-z_i)t_i$ relevant for $\$_B(\theta)$ are not ordered, and, in fact, $\$_B(\theta)$ is typically dominated by a sequence of emissions imparting roughly equal k_T and is therefore considerably more influenced by incalculable large distance effects than $\$_F(\theta)$.

To $O(\alpha_s)$, $e^+e^- \to q\bar{q}G$ spreads the $q,\bar{q}$ jets and modifies the $O(\alpha_s^0)$ result $\$_F(\theta) = 1$ to $\$_F(\theta) \simeq 1 - \int_{\theta^2 s}^{s} \frac{dt}{t}\frac{\alpha_s(t)}{2\pi}\int_0^1 z(1-z)P_{qG}(z)dz \simeq 1 + \frac{2\alpha_s}{\pi}\log\theta$ in the LLA [F.10]. (For a two-gluon-jet final state, this becomes $\$_F(\theta) \simeq 1 + (\frac{42+F}{10})\frac{\alpha_s}{\pi}\log\theta$.) In higher orders, $\$_F(\theta)$ may be computed as the mean product of the absolute fractional energies ($\langle z_i^{abs} z_{i'}^{abs}\rangle = \langle \prod_{j=1}^{i-1} z_j^2(1-z_i)\rangle$) summed over all pairs of emitted partons with $t \geq \theta^2 s$. In calculating the contribution of the i^{th} emission, the z_j for $j < i$ obey $0 \leq z_j \leq 1$, and the virtual diagrams at $z_j = 1$ entirely cancel the soft divergences. (This is in contrast to the case of $\$_B(\theta)$, where $(1-z_j) \leq \theta^2 s/t_j$, thus leaving uncanceled a $\log(t_j/\theta^2 s)$ term from the soft emission region.) The t_j integrals are, however, restricted according to $t_j \geq t_{j+1} \ldots\ldots \geq t_i \geq \theta^2 s$. In performing the t_j integrations, one must retain the variation of $\alpha_s(t_j) \sim 1/\log(t_j/\Lambda^2)$, leading to $[\log(\log(\theta^2 s/\Lambda^2)/\log(s/\Lambda^2))]^k/k!$ terms at each order. Then, summing over all possible jet backbones and numbers of emissions, one obtains the exponentiated form

$$\$_F(\theta) \simeq \frac{T(\theta)}{\pi}\left[\int_0^1 P(z)z(1-z)dz\right]\cdot\exp\left[-\int_0^1 P(z)z^2dz\right.$$

$$\left.\log(T(\theta))(6/(33-2F))\right]\cdot I, \tag{7}$$

$$T(\theta) \simeq \frac{\alpha_s(s)}{\alpha_s(s\theta^2)} \simeq \frac{\log(s\theta^2/\Lambda^2)}{\log(s/\Lambda^2)}$$

where P is the matrix of kernels P_{ij}, and I is a vector in (q,G) space representing the initial parton. Hence, for quark and gluon jets [12,1]

$$\begin{aligned}(\$_F(\theta))_q &\simeq 1.2[T(\theta)]^{0.6} - 0.2[T(\theta)]^{1.4}\\ (\$_F(\theta))_G &\simeq 0.4[T(\theta)]^{0.6} + 0.6[T(\theta)]^{1.4}.\end{aligned} \tag{8}$$

Without knowledge of subleading log terms, one cannot determine the optimal argument of α_s (or T) to be used in applications of these formulae to jets produced in specific processes; plausible choices give rather different phenomenological estimates for spreading of jets. (From eq. (8), one may estimate $\langle H_\ell\rangle$ for $\zeta \to GGG...$ at large ℓ. The lowest-order process has a differential cross-section barely distinguishable from three-body phase space and gives $\langle H_\ell\rangle \simeq 3/8$; higher order processes serve simply to multiply this by $\simeq (\$_F(1/\ell))_G$. Note that at high ℓ, the $\langle H_\ell\rangle$ for this 3-jet process $\sim \log\ell$, whereas for two jet processes such as $e^+e^- \to q\bar{q}....$, $\langle H_\ell\rangle \sim \log^2\ell$.)

Most of the radiation in a jet consists of soft partons. One may estimate the multiplicity of partons with absolute fractional energies $E/\sqrt{s}$ above some small cutoff x_{min} by integrating the differential cross-section (1) with the restrictions $x_{min}/(\prod_{j=1}^{i-1} z_j) \le z_i \le 1$ and summing over all possible jet backbones. Consider first the emission of gluons in a gluon jet, so that the

z_i integrands are roughly $c_A/\pi\ 1/z_i$ (the multiplicity is dominated by soft gluons emitting soft gluons). The nested lower limits on the z_i integrals result in a triangular integration region (analogous to that for the t_i), and for k gluon emission gives $[(c_A/\pi)\log x_{min}]^k/k!$; the corresponding t integrals give a factor $[\log\log(s/\Lambda^2)]^k/k!$. The terms from k gluon emission therefore $\sim A^k/(k!)^2$. The sum over k may be performed by recalling the expansion of irregular Bessel functions: $I_n(2y) = \sum_{k=0}^{\infty} y^{2k+n}/(k!(k+n)!)$; their asymptotic expansion is $I_n(y) \sim e^y/\sqrt{2\pi y}$. To obtain a complete result, one must include the $O(1)$ as well as $O(1/z)$ parts of $P(z)$: such terms give no $\log(z)$ contributions and exponentiate to a power of α_s [F.11]. Summing over all possible emissions, one estimates that the number of gluons with fractional energies $\gtrsim x_{min}$ in a gluon jet is (taking $F = 3$ and $t_c = \Lambda^2$)

$$\begin{aligned} \langle n_G\rangle_G &\simeq I_0(2\sqrt{A})[\alpha_s(s)]^{1.25} \\ A &\simeq \frac{c_A}{\pi}\log(\alpha_s(s))\log(x_{min}). \end{aligned} \tag{9}$$

In a quark jet, the probability for the first gluon emission is reduced by a factor $C_F/C_A = 4/9$, but the subsequent development remains the same, so that the number of gluons in (9) is just reduced by 9/4. Soft quarks emitted from gluons follow a dz rather than dz/z spectrum; most light quarks at small z thus arise from a series of gluon emissions followed by a single materialization $G \to q\bar{q}$, so that the last $2C_A\log(z)$ for gluon emissions is replaced by just F/2. Then the multiplicity of quarks with energies $\gtrsim x_{min}$ in a gluon jet becomes

$$\langle n_q\rangle_G \simeq \frac{F}{2\pi}\log(1/\alpha_s(s))I_1(2\sqrt{A})/\sqrt{A}[\alpha_s(s)]^{1.25}. \tag{10}$$

The x distributions of soft partons in a jet may be found by differentiating (9) and (10) or directly by not integrating over the last emission in the construction of the series (or, alternatively, by inverting the behavior of z^{n-1} moments due to $O(\frac{1}{n-1})$ and $O(1)$ terms in the anomalous dimensions [7,12]).

Perturbation theory presumably ceases to be operative when the invariant masses of partons in a jet fall below $\sim \sqrt{t_c}$, but some properties of the parton system prepared may be relevant for subsequent condensation into hadrons. One of these features is the invariant mass distribution for pairs of final partons (each with $t \leqslant t_c$) in the jet [13]. Such partons may be taken as emitted from the backbone of the jet, which consists of a sequence of radiating partons with large t. The invariant mass of the k^{th} and $(k+1)^{th}$ real partons emitted is $M^2 \simeq (1-z_{k+1})t_k$. In computing the mean number of such pairs with $M^2 \geqslant M_o^2 >> t_c \sim \Lambda^2$, the corresponding limits on the z_i are $0 \leq z_i \leq 1$ $(i \leq k)$, $0 \leqslant z_{k+1} \leqslant 1 - M_o^2/t_i$, while the t_i satisfy $M_o^2 \leqslant t_i \leqslant t_{i-1}$ $(i \leq k)$, $t_c \leqslant t_{k+1} \leqslant t_k$. Performing the z_{k+1} integral introduces a crucial $\log(t_k/M_o^2)$. In all t_i integrals, the variation of this term overwhelms the running of $\alpha_s(t)$ in the LLA and prevents the appearance of log log(t) term. Instead, the final result $\sim \log^k(s/M_o^2)/k!$. Summing this over the position of the pair and dividing by the total number of pairs (i.e., $M_o^2 = t$), one obtains for the probability that a given pair has $M > M_o$ the power-law damped form $\sim (M^2/t_c)^{-\alpha_s\gamma}$, where γ depends on the types of partons in the pair and jet. (Note that in an asymptotically-free theory such as ϕ_6^3 with no soft divergences, the $\log(t_k/M_o^2)$ from the z_{k+1} integral is absent, and the spectrum $\sim [\log(M^2/t_c)]^{-p}$. Also note that the damped spectrum is independent of the color of the pair; for a sequence of n produced partons $M^2 \sim (1-z_{k+1}\dots z_{k+n-1})t_k$.) If instead of considering a pair of 'final' partons each with $t \leqslant t_c$, one allows one parton in the pair to have arbitrary mass, then the pair mass spectrum is just $\not{s}_B(M^2/s)$ and is only logarithmically damped.

External forces acting on a sufficiently small color singlet system of partons should cancel coherently, so that its later evolution is independent of the rest of the final state. The argument of the previous paragraph suggests that at a time $\sim 1/\sqrt{t_c}$, the invariant mass of a nearby pair of partons is peaked around $\sqrt{t_c} \sim 1$ GeV. It is therefore plausible that when such pairs constitute color singlet systems, they should condense directly into clusters of hadrons, probably isotropically in their rest frames. The relevant pairings are perhaps chosen according to the spatial separation of the final partons: A convenient and largely equivalent picture is that every parton trails a 'string' representing each spinor color index (hence two strings per gluon), and that it is the strings which eventually form hadrons. This picture implies that the ultimate fragmentation of gluon jets should be like pairs of quark jets and requires no further parameters. (Equations (9) and (10) support this when $N_c \to \infty$ so that $C_A/C_F \to 2$.) An alternative method would be to ignore the colors of partons and fragment each separately to hadrons when its t reaches $t_o \gg \Lambda^2$, using a phenomenological model fit at $s \simeq t_o$; predictions should be independent of t_o. The latter method is commonly applied to deduce the dependence of single hadron momentum spectra on s. For complete final states it is more difficult to implement: A Monte Carlo model based on the former method will be described in [14].

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Footnotes

[F.1] In the deep inelastic scattering cross-section (σ), non-kinematical $O(\Lambda^2/Q^2)$ effects presumably arise from rescattering of the struck quark at large times. In a heavy nucleus, the effect of this rescattering $\sim A^{1/3}$. Thus, $O(\Lambda^2/Q^2)$ terms in σ should behave $\sim A^{4/3}$, while scaling (up to short-distance QCD corrections) terms should $\sim A^1$. This fact may allow extrapolations to obtain better estimates of the latter at small Q^2.

[F.2] The relevant scale for the variation of α_s is determined by the $O(\alpha_s^2)$ result [3] $1 + \alpha_s(s)/\pi + (2.0-0.1\ F)(\alpha_s/\pi)^2$, where α_s (or Λ) is defined to be extracted from measurements on another process using theoretical predictions calculated in the truncated minimal subtraction renormalization scheme, with $Tr[1] = 4$.

[F.3] This behavior (essentially kinematic in origin) is manifest when mass corrections are computed. For example, $\langle H_\ell \rangle$ or $\langle \text{thrust} \rangle$ typically contain $O(\sqrt{m^2/s})$ corrections, which are forbidden for σ by power-counting theorems for the corresponding Feynman diagrams.

[F.4] The appearance of these soft divergences is specific to vector field theories; they do not occur with scalar gluons.

[F.5] The kernels $P_{ij}(z)$ which represent the probability (in units of $\alpha_s/2\pi t$) that parton i will emit parton j carrying a fraction of its energy (strictly, longitudinal Sudakov variable) are given by [6,7]

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

$$P_{qG}(z) = C_F\left(\frac{1+(1-z)^2}{z}\right) = P_{qq}(1-z)$$

$P_{Gq}(z) = \frac{F}{2}(z^2+(1-z)^2)$ (summing over F flavors of light quarks)

$$P_{GG}(z) = 2C_A\left(\frac{(1-z+z^2)^2}{z(1-z)}\right)_+ - \frac{F}{3}\delta(1-z)$$

where the color factors $C_A = N_C = 3$, $C_F = (N_C^2-1)/(2N_C) = 4/3$.

[F.6] The details of this derivation depend on the gauge used. $P_{qq}(z) \sim 1/(1-z)$ when n is approximately along the q ($\bar{q}$) direction, so that only radiation from the $\bar{q}$ (q) gives leading logs; otherwise $P_{qq} \sim 1/(1-z+t/s)$ ($\sim 1/x_G$) but both q and $\bar{q}$ radiate. The former approach is used here; in the latter, the t_1 integral becomes $\int_{\theta^2 s}^{s} \frac{dt_1}{t_1}\log(\theta^2 s/t_1 + t_1/s) \simeq \frac{1}{2}\int_{\theta^2 s}^{s}\frac{dt_1}{t_1}\log(\theta^2 s/t_1)$, thus compensating for the different number of contributing diagrams. (z is defined as the relative Sudakov variable; other choices differ by O(t/s), but give different phase space boundaries.)

[F.7] Defining the differential energy correlation $F_2^{pt}(\chi) = \sum_{\text{partons}} 2E_iE_j/s\ \delta(\cos\phi_{ij}-\chi)$ (so that $H_\ell = \int_{-1}^{1} F_2^{pt}(\chi)P_\ell(\chi)/2\, d\chi$), the integral (which coincides with the previous definition of $\not{s}_B(-\cos^{-1}(\eta))$ to leading log order)

$$\int_\eta^1 F_2^{pt}(\chi)d\chi \simeq 1 - \frac{2\alpha_s}{3\pi}\left[\log^2\left(\frac{1+\eta}{2}\right) + 3\log\left(\frac{1+\eta}{2}\right) + 4.7 + \ldots\right].$$

[F.8] Again, details of derivation depend on gauge. The exponential form has been verified explicitly to $O(\alpha_s^2)$ in [8] (the more complicated terms found in [9] using the second gauge in [F.6] appear to be absent). For a final state of two gluon jets (e.g., from a ${}^1S_o$, ${}^3P_o$ or ${}^3P_2$ $Q\bar{Q}$ state), the exponent here is multiplied by $C_A/C_F = 9/4$.

[F.9] In analogy with [F.8], off shell quarks give $P_{qq}(z) \sim 1/(1-z+p^2/s)$, but off shell gluons leave $P_{qq}(z) \sim 1/(1-z)$. The coefficient of $\log^2(p^2/s)$ in the Sudakov form factor for off-shell q is thus 1/2 that for off-shell G.

[F.10] In this case, the quantity defined in [F.5] becomes $\int_\eta^1 F_2^{pt}(\chi)d\chi \simeq 1 + \frac{\alpha_s}{\pi}\,[\log(1-\eta) - 0.40 + \ldots]$.

[F.11] To see this, first sum over the number of $O(1/z)$ kernels with a fixed set of $O(1)$ kernels. Note that the effects of the $O(1)$ kernels are of the same order as those of subleading log terms in the cross-section and therefore can only be considered indicative of such corrections.

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